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THE FLOW OF VISCOPLASTIC MATERIAL BETWEEN TWO CONCENTRIC SPHERES

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The motion of a viscoplastic medium between two concentric spheres is considered upon rotation of one sphere with constant angular velocity. This problem is solved by an heuristic iterative method. The boundary of the stagnation zones is found and its specific shape is shown. The flow characteristics versus the parameter of the medium are obtained.

Many studies are devoted to the rotation of bodies in a viscoplastic medium. The problem of steady rotation of the simplest axisymmetrical bodies (the cylinder, cone, and sphere) in an unbounded medium was considered in [1–5]. Using the approximations adopted in the boundary-layer theory, Kolbovskii [6] studied the stationary rotation of an arbitrary convex axisymmetrical body. Chernyshev [7] considered the flow of a viscoplastic medium between two coaxial cones. The flow problem of a viscoplastic medium adjacent to a cylinder rotating with variable angular velocity was solved by Safronchik [8].

In the present work, the motion of a viscoplastic medium between two concentric spheres is considered upon rotation of one of the spheres with constant angular velocity. This rotation occurs, for example, in rotational viscosimeters to study viscoplastic materials.

The variational [9, 10] and heuristic [11] methods are known to be suitable for solving the viscoplastic-flow problem. The application of the variational method in this case is quite effective; however, in the presence of unknown boundaries such as the stagnation zones, one faces some difficulties. Therefore, the heuristic method is applied to solve the posed problem. The advantage of this method over other methods of numerical solution is that the boundary-value problem of one ordinary differential equation of the second kind is solved at each step of the iterative process. The velocity field which is used to find the flow characteristics is obtained. The boundary of the stagnation zones is found at various values of the Saint-Venant parameter and its specific shape is shown.

1. Formulation of the Problem. The steady motion of an incompressible viscoplastic medium between two concentric spheres is considered in the absence of external forces in a spherical coordinate system. The external sphere of radius R_2 is at rest, and the internal sphere of radius R_1 rotates with constant angular velocity ω_0.

In the case of body rotation, only one component of the flow rate of the medium $V_\varphi = V(r,\theta)$ is different from zero. The equation of state of a viscoplastic medium in the viscous-flow region has the form [12]

$$\sigma_{ij} = \Big(2\mu + \frac{\sqrt{2}\tau_0}{(\varepsilon_{kl}\varepsilon_{kl})^{1/2}}\Big)\varepsilon_{ij} - p\delta_{ij} \qquad (p = -\sigma_{ii}/3)$$

where σ_{ij} is the stress tensor, ε_{ij} is the strain-rate tensor, μ is the coefficient of viscosity, and τ_0 is the yield point.

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In the case considered, the components of the strain-rate and stress tensors are not zero:

$$\varepsilon_{\theta\varphi}=\frac{1}{2}\Big(\frac{1}{r}\frac{\partial V}{\partial\theta}-\frac{\cot\theta}{r}V\Big),\quad \varepsilon_{r\varphi}=\frac{1}{2}\Big(\frac{\partial V}{\partial r}-\frac{V}{r}\Big),\quad \sigma_{\theta\varphi}=\Big(2\mu+\frac{\tau_0}{(\varepsilon_{\theta\varphi}^2+\varepsilon_{r\varphi}^2)^{1/2}}\Big)\varepsilon_{\theta\varphi},$$
$$\sigma_{r\varphi}=\Big(2\mu+\frac{\tau_0}{(\varepsilon_{\theta\varphi}^2+\varepsilon_{r\varphi}^2)^{1/2}}\Big)\varepsilon_{r\varphi},\qquad \sigma_{rr}=\sigma_{\varphi\varphi}=\sigma_{\theta\theta}=-p. \tag{1.1}$$

The flow rate of the medium V is connected with the angular velocity ω by the relation $V=r\omega(r,\theta)\sin\theta$, which allows one to rewrite formulas (1.1) in the form

$$\varepsilon_{\theta\varphi}=\omega_\theta\frac{\sin\theta}{2},\quad \varepsilon_{r\varphi}=r\omega_r\frac{\sin\theta}{2},\quad \sigma_{\theta\varphi}=\Big(\mu\sin\theta+\frac{\tau_0}{(\omega_\theta^2+r^2\omega_r^2)^{1/2}}\Big)\omega_\theta,$$
$$\sigma_{r\varphi}=\Big(\mu\sin\theta+\frac{\tau_0}{(\omega_\theta^2+r^2\omega_r^2)^{1/2}}\Big)r\omega_r,\qquad \sigma_{rr}=\sigma_{\varphi\varphi}=\sigma_{\theta\theta}=-p. \tag{1.2}$$

We write the equations of equilibrium of a viscoplastic medium in the spherical coordinate system:

$$\frac{\partial\sigma_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\varphi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}+\frac{1}{r}(3\sigma_{r\varphi}+2\sigma_{\theta\varphi}\cot\theta)=0,\quad \frac{\partial p}{\partial r}=0,\quad \frac{\partial p}{\partial\theta}=0.$$

It follows that $p=p(\varphi)$ and $dp/d\varphi=\text{const}$. We obtain $dp/d\varphi=0$ from the condition of pressure periodicity $p(\varphi)=p(\varphi+2\pi)$, and hence

$$\frac{\partial\sigma_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\varphi}}{\partial\theta}+\frac{1}{r}(3\sigma_{r\varphi}+2\sigma_{\theta\varphi}\cot\theta)=0. \tag{1.3}$$

Substituting the values of the stress-tensor components (1.2) into (1.3), we obtain the following differential equation of motion of an incompressible viscoplastic medium in the spherical coordinates:

$$\mu\Big(r\omega_{rr}+4\omega_r+\frac{1}{r}\omega_{\theta\theta}+\frac{3\omega_\theta\cot\theta}{r}\Big)$$
$$+\frac{\tau_0}{\sin\theta}\Big(\frac{r\omega_\theta^2\omega_{rr}+4\omega_r\omega_\theta^2+3r^2\omega_r^3+r\omega_r^2\omega_{\theta\theta}-2r\omega_r\omega_\theta\omega_{r\theta}}{(\omega_\theta^2+r^2\omega_r^2)^{3/2}}+\frac{2\omega_\theta\cot\theta}{r(\omega_\theta^2+r^2\omega_r^2)^{1/2}}\Big)=0. \tag{1.4}$$

This equation holds true only in the viscous-flow region inside which the condition $\operatorname{grad}V\neq 0$ is satisfied.

If the internal sphere is rotated with small angular velocity, it will involve in motion a small part of the viscoplastic medium whose external boundary Γ is unknown beforehand. The internal boundary Γ_0 in the flow region is the surface of the rotating sphere; the external boundary Γ is of complex form and should be found during solution of the problem. The medium behaves like a rigid body beyond the boundary, and this region is called a stagnation zone.

At the points which belong to the stagnation zones, the equality

$$\operatorname{grad}V=0 \tag{1.5}$$

holds true. For the points lying at the boundary of the stagnation zones, condition (1.5) can be written as follows [13]:

$$\frac{\partial V}{\partial n}=0,$$

where n is the normal to the boundary of the stagnation zones.

Taking into account the aforesaid and the physical pattern of the phenomenon, we obtain the following boundary conditions:

$$\omega\Big|_{\Gamma_0}=\omega_0,\qquad \omega\Big|_{\Gamma}=0,\qquad \frac{\partial\omega}{\partial n}\Big|_{\Gamma}=0. \tag{1.6}$$

Thus, the function ω will be the solution of the boundary-value problem (1.4), (1.6) for a nonlinear partial differential equation.

We pass to the dimensionless variables. For this purpose, the angular velocity is referred to the quantity ω_0, and the linear sizes to the radius of the internal sphere R_1.

In dimensionless variables, the boundary-value problem (1.4), (1.6) takes the form

$$r\omega_{rr} + 4\omega_r + \frac{1}{r}\omega_{\theta\theta} + \frac{3\omega_\theta \cot\theta}{r}$$
$$+\frac{1}{H\sin\theta}\Big(\frac{r\omega_\theta^2\omega_{rr} + 4\omega_r\omega_\theta^2 + 3r^2\omega_r^3 + r\omega_r^2\omega_{\theta\theta} - 2r\omega_r\omega_\theta\omega_{r\theta}}{(\omega_\theta^2 + r^2\omega_r^2)^{3/2}} + \frac{2\omega_\theta\cot\theta}{r(\omega_\theta^2 + r^2\omega_r^2)^{1/2}}\Big) = 0; \tag{1.7}$$

$$\omega\Big|_{\Gamma_0} = 1, \qquad \omega\Big|_{\Gamma} = 0, \qquad \frac{\partial\omega}{\partial n}\Big|_{\Gamma} = 0, \tag{1.8}$$

where $H = \mu\omega_0/\tau_0$ is a dimensionless parameter. The quantity $S = H^{-1}$ is called the Saint-Venant parameter.

2. Method of Solution. To solve the boundary-value problem (1.7), (1.8), we use the iterative method [11]. We describe it briefly. In Eq. (1.7), we pass from variable (r, θ) to variable (ξ, η) by making the following replacement:

$$\xi = \xi(r,\theta), \qquad \eta = \eta(r,\theta). \tag{2.1}$$

In introducing the variable ξ, one can ignore the derivatives of ω with respect to η and the mixed derivatives compared with other terms, because the variable ξ is chosen, according to the method of [11], in a way such that the variation in ω along the line ξ = const should be small. Then the function ω will be the solution of the boundary-value problem for the second-order ordinary linear differential equation

$$(\xi_\theta^2 + r^2\xi_r^2)\omega_{\xi\xi} + (r^2\xi_{rr} + 4r\xi_r + \xi_{\theta\theta} + 3\xi_\theta\cot\theta)\omega_\xi + \frac{1}{H\sin\theta}((r^2\xi_\theta^2\xi_{rr} + 4r\xi_r\xi_\theta^2 + 3r^3\xi_r^3$$
$$+r^2\xi_r^2\xi_{\theta\theta} - 2r^2\xi_r\xi_\theta\xi_{r\theta})(\xi_\theta^2 + r^2\xi_r^2)^{-3/2} + 2\xi_\theta\cot\theta(\xi_\theta^2 + r^2\xi_r^2)^{-1/2}) = 0; \tag{2.2}$$

$$\omega\Big|_{\xi=1} = 1, \qquad \omega\Big|_{\xi=0} = 0. \tag{2.3}$$

The variable η plays the role of a parameter in the integration of Eq. (2.2). The zero approximation in (2.1) is denoted by ξ^0. Having solved problem (2.2), (2.3), we find $\omega^0(\xi^0,\eta)$. Having denoted $\omega^0(\xi^0,\eta)$ by ξ^1, to determine the following approximation we solve a boundary-value problem similar to problem (2.2), (2.3) with variables (ξ^1,η). We act similarly in subsequent iterations.

The most easily implemented iterative process is obtained if one assumes that $\eta = \theta$ at each step. With variable η chosen in this way, after the inverse transition from variable (ξ, η) to variable (r, θ) Eq. (2.2) remains a linear differential equation:

$$\omega_{rr} + \Big(4r\xi_r^2 - \frac{\xi_{rr}\xi_\theta^2}{\xi_r} + \xi_r\xi_{\theta\theta} + 3\xi_r\xi_\theta\cot\theta\Big)(\xi_\theta^2 + r^2\xi_r^2)^{-1}\omega_r + \frac{1}{H\sin\theta}((r^2\xi_\theta^2\xi_{rr} + 4r\xi_r\xi_\theta^2 + 3r^3\xi_r^3$$
$$+r^2\xi_r^2\xi_{\theta\theta} - 2r^2\xi_r\xi_\theta\xi_{r\theta})(\xi_\theta^2 + r^2\xi_r^2)^{-3/2} + 2\xi_\theta\cot\theta(\xi_\theta^2 + r^2\xi_r^2)^{-1/2})\xi_r^2(\xi_\theta^2 + r^2\xi_r^2)^{-1} = 0; \tag{2.4}$$

$$\omega\Big|_{\Gamma_0} = 1, \qquad \omega\Big|_{\Gamma} = 0. \tag{2.5}$$

The variable θ enters into (2.4) as a parameter.

The problem is solved according to a scheme which is based on the use of the Aytekin method [14] and improves the convergence of the iterative process. The iterative method proposed in [11] was applied to the solution of the problems of antiplane deformation of a viscoplastic medium under the conditions of pure shear realized in [15].

Although the method described in the present work is used for solving a different class of problems than those studied in [15], the algorithms of their realization are very similar, because they are based on the technique suggested in [11]. However, there are some differences in the implementation of these methods. In particular, the boundary-value problem at fixed values of θ was solved by the ballistic method rather than by

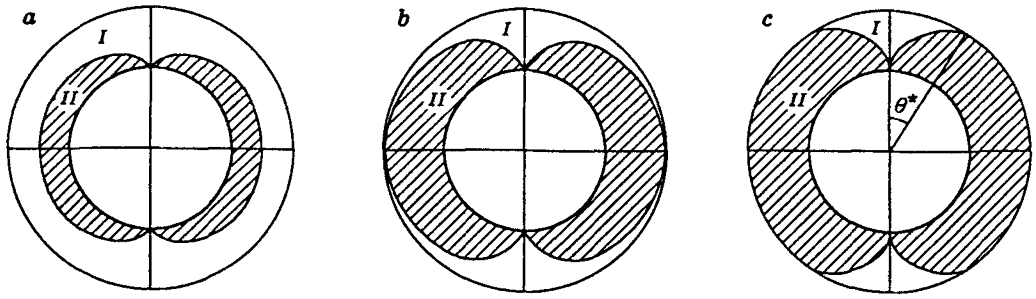

Fig. 1

reducing it to two Cauchy problems [15]. In addition, the shape of the stagnation-zone boundary is determined differently. We shall describe this below.

3. Flow of the Medium Between Two Concentric Spheres. The described method is used to solve the boundary-value problem (2.4), (2.5). To find the zero approximation, we set

$$\xi^0 = \frac{R_* - r}{R_* - R_1}, \qquad \eta = \theta,$$

where $r = R_*(\theta)$ is the contour equation of the external boundary Γ of the flow region.

As mentioned above, the shape of the boundary which separates the stagnation zone and the flow region is not known beforehand. We determine its initial shape by the following technique. Setting $\omega_\theta = \omega_{\theta\theta} = \omega_{r\theta} = 0$ in Eq. (1.7), we obtain an equation with the boundary conditions

$$\omega_{rr} + \frac{4}{r}\omega_r - \frac{3}{Hr^2\sin\theta} = 0, \quad \omega\Big|_{r=R_1} = 1, \quad \omega\Big|_{r=R_*} = 0, \quad \frac{\partial\omega}{\partial r}\Big|_{r=R_*} = 0.$$

Solving this problem by the method of variation of arbitrary constants, we obtain the formula of the velocity distribution

$$\omega = \frac{1}{3H\sin\theta}\Big(\frac{R_*^3}{r^3} + 3\ln\frac{r}{R_*} - 1\Big),$$

where the radius of the flow-propagation zone R_* is the solution of the transcendental equation

$$\frac{R_*^3}{R_1} - 3\ln\frac{R_*}{R_1} = 1 + 3H\sin\theta. \tag{3.1}$$

One can use R_*, i.e., the solution of (3.1), as the initial shape of the contour Γ which bounds the flow region from outside. The condition $\omega_r < 0$ should be satisfied inside the flow region. For fixed θ, problem (2.4), (2.5) is solved by the ballistic method.

If the flow partially reaches the boundary of the external sphere, one needs to solve a new boundary-value problem without the condition that the slip velocities vanish on the corresponding part of the boundary of the external sphere to determine the velocity distribution. After this, it is necessary to check the satisfaction of the condition Γ at the boundary of the flow region $\partial\omega/\partial r = 0$ and deform its contour to refine the boundary shape.

The moment of friction forces acting on the surface of the internal sphere is equal to

$$M = 2\pi\int_0^\pi \sigma_{r\varphi}\Big|_{\Gamma_0} R_1^3\sin^2\theta\,d\theta,$$

where $\sigma_{r\varphi} = (1/(\omega_\theta^2 + r^2\omega_r^2)^{1/2} + H\sin\theta)r\omega_r$ is the tangent stress.

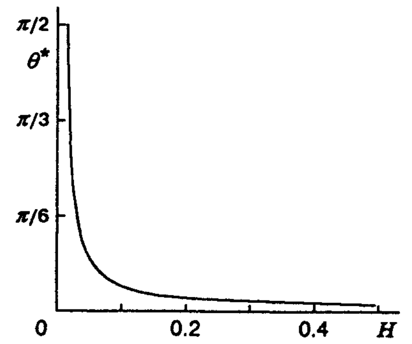

Fig. 2

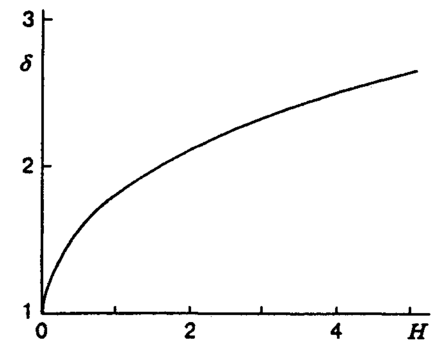

Fig. 3

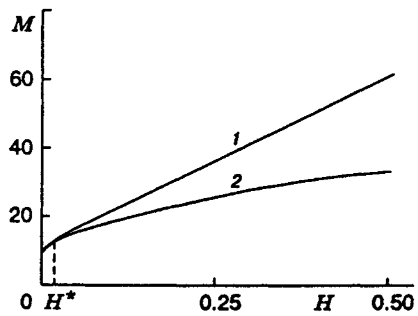

Fig. 4

This solution allows one to draw the following conclusion. The shape of the surface of stagnation zone I qualitatively depends on the magnitude of the Saint-Venant parameter H. For example, for small angular velocities, when H is smaller than a certain value H^*, flow region II of the medium is located between the internal sphere and the stagnation zone, which does not touch the external sphere (Fig. 1a). Outside this region, the medium is in a rigid unstrained state. As the value of H increases, the stagnation-zone boundary approaches the external sphere and, for $H = H^*$, touches the external sphere on the equator for the first time (Fig. 1b). For $H > H^*$, the stagnation zone is divided into "northern" and "southern" symmetrical parts (Fig. 1c). The boundary of each part is determined by the angle θ^*.

Figure 2 shows the angle θ^* versus the parameter H. With increase in H, the angle θ^* decreases, i.e., the dimensions of the stagnation zone are decreased. It is noteworthy that, for any H, the stagnation zone is always adjacent to the poles of the external sphere and is shaped like a cone facing the internal sphere from the external to the internal sphere. The vertex of this cone touches the pole of the internal sphere (Fig. 1c). Thus, in this case the stagnation zone always exists for any finite value of H in the neighborhood of the poles of the external fixed sphere.

If the external sphere is absent, we arrive at the problem of sphere rotation in an unbounded viscoplastic medium. Figure 3 shows the maximum radius of the flow region δ in the equatorial plane, i.e., for $\theta^* = \pi/2$, as a function of H.

Figure 4 shows the dependence of the moment M applied to the internal sphere on H for the case of a bounded and unbounded medium (curves 1 and 2, respectively). For $H \leqslant H^*$, these curves coincide; for $H > H^*$, the moment of friction in the presence of the external sphere is greater than that in the case of sphere rotation in an unbounded medium.

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